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DOUBLE ELLIPTIC GEOMETRY IN TERMS OF POINT AND ORDER ALONE.[*]

BY J. R. KLINE.

1. Introduction.

In his Rational Geometry,[†] Halsted built up two-dimensional double elliptic geometry, in terms of the undefined symbols point, order, association and congruence. In the present paper I propose a categorical set of ten mutually independent axioms for three-dimensional double elliptic geometry. Only the symbols point and order are undefined.

I wish to express my deep gratitude to Professor Robert L. Moore, who suggested the problem of this paper and aided me continually in its preparation.

2. Axioms and Definitions.

AXIOM I. If A is a point, then there exists at least one point A', different from A, such that $AA'C$[‡] is false for every C.

DEFINITION 1. If A and B are distinct points such that ABC is false for every C, then B is said to be an *opposite* of A. If A is a point, A' denotes an opposite of A.

AXIOM II. There exist two distinct points A and B such that B is not an opposite of A.

AXIOM III. If ABC, then CBA.

AXIOM IV. If ABC and A' is an opposite of A, then $A'CB$.

AXIOM V. If ABC and ACD, then ABD.

DEFINITION 2. The *line* $AB(A \neq B \neq A')$ is the set of all points $[X]$ such that $X = A$, $X = A'$, $X = B$, $X = B'$, XAB, AXB, ABX or $AX'B$. The set of all points $[X]$ such that AXB constitute the *segment* AB. A and B are the *end points* of this segment.

AXIOM VI. If there exist three distinct points, no two of which are opposites, then there exist three distinct points A, B and C such that A is not an opposite of B and such that C is not on the line AB.

AXIOM VII. If D is not on the line AB and C and E are such that ABC and BED and if there exists F such that DFA, then there exists G such that CEG and DGA.

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[*] Presented to the Society, April 24, 1915.

[†] G. B. Halsted, Rational Geometry, John Wiley and Sons, pp. 212–247.

[‡] The abbreviation "$AA'C$" signifies that the points A, A' and C are in the order $AA'C$.

Definition 3.* Three non-collinear points A, B, C are the *vertices* of a *triangle* ABC, whose *sides* are the segments AB, BC, CA and whose *boundaries* are its vertices and its sides.

Definition 4. The *interior* of a triangle ABC is the set of all points $[X]$ such that X lies on a segment whose end points are on different sides of the triangle ABC.

Definition 5. If A, B and C are non-collinear points, the *plane* ABC consists of all points collinear with two points of the sides of the triangle ABC.

Axiom VIII. If there exist three distinct non-collinear points, then there exists a plane ABC and a point D such that D is not on the plane ABC.

Definition 6. If A, B, C and D are four non-collinear points, they are the *vertices* of a *tetrahedron* $ABCD$, whose *faces* are the interiors of the triangles ABC, BCD, CDA, DAB, whose *edges* are the segments AB, BC, CD, DA, AC and BD.

Definition 7. If A, B, C and D are the vertices of a tetrahedron, the *space* $ABCD$ consists of all points collinear with two points of the faces of the tetrahedron $ABCD$.

Axiom IX. If there exist four distinct points, neither lying on the same line, nor lying in the same plane, then there exists a space $ABCD$ such that there is no point not collinear with two points of the space $ABCD$.

Axiom X. If all the points of a segment are divided into two mutually exclusive classes $[S_1]$ and $[S_2]$ such that no point of S_1 lies between two points of S_2 and such that no point of S_2 lies between two points of S_1, then there exists H such that if, A is a point of $S_1 \dotplus H$ and B is a point of $S_2 \dotplus H$, then AHB.

3. Consequences of Axioms I–V.

Theorem I. If A and B are two distinct points and B is not an opposite of A, then there exists C such that ABC.

Theorem 2. If A_1 is an opposite of A, then A is an opposite of A_1.

Proof. Suppose A is not an opposite of A_1. Then, by Theorem 1, there exists C such that A_1AC. By Axioms III, I, IV and III, A_1AC implies AA_1C'. But AA_1C' is contrary to hypothesis. Hence A is an opposite of A_1.

Theorem 3. If ABC, then C is different from A, B, every opposite of A and every opposite of B.

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* Definitions 3, 4, 5, 6 and 7 and Axioms VIII and IX are the same as Definitions 2, 5, 6, 7 and 8 and Axioms IX and X of Veblen's System of Axioms for Geometry, Transactions of the American Mathematical Society, vol. 5 (1904), page 345.

Case I.—Suppose $C = A$. Then ABA. By Axioms IV, III, IV and III, ABA implies $AA'B'$. But $AA'B'$ is contrary to the Definition of A'. Hence $C \neq A$.[*]

Case II. Suppose $C = A'$. Then ABA'. By Axiom IV, ABA' implies $A'A'B$. By Axiom III, ABA' implies $A'BA$. By Axiom V, $A'A'B$ and $A'BA$ imply $A'A'A$. By Axiom III, $A'A'A$ implies $AA'A'$, which is contrary to the definition of A'. Hence $C \neq A'$.

Case III. Suppose $C = B$. Then ABB. By Axioms III, IV and III, ABB implies BAB'. But BAB' is contrary to Case II. Hence $C \neq B$.

Case IV. Suppose $C = B'$. Then ABB'. By Axioms IV and III, ABB' implies $BB'A'$, which is contrary to the Definition of B'. Hence $C \neq B'$.

Theorem 4. If ABC, then not BCA.[†]

Proof. Suppose ABC and BCA. By Axiom III, BCA implies ACB. By Axiom V, ABC and ACB imply ABB. But ABB is contrary to Theorem 3. Hence not BCA.

Theorem 5. If two distinct points are opposites of the same point, then they are opposites of each other.

Proof. Suppose the two distinct points A_1 and A_2 are both opposites of A. Then by Theorem 2, A is an opposite of A_1 and of A_2. Suppose A_2 is not an opposite of A_1. Then, by Theorem 1, there exists C such that A_1A_2C. By Axioms IV, III, IV and III, A_1A_2C implies CAA. But CAA is contrary to Theorem 3. Hence A_2 is an opposite of A_1. In like manner A_1 is an opposite of A_2.

Theorem 6. If X is any point, then there exists a point Y different from X and not an opposite of X.

Proof. Suppose there is no such point Y. Then every point either coincides with X or is an opposite of X. By Axiom II, there exist two distinct points A and B such that B is not an opposite of A.

Of the following four conceivable cases, the first two are evidently impossible.

Case I. A is X and B is X.

Case II. A is X and B is an opposite of X.

Case III. B is X and A is an opposite of X, consequently of B. Then, by Theorem 2, B is an opposite of A, which is contrary to hypothesis.

Case IV. A and B are both opposites of X. Then, by Theorem 5, B is an opposite of A, which is contrary to hypothesis.

Theorem 7. No point has more than one opposite.

Proof. Suppose the point C has two opposites C_1 and C_2. By Theorem 6, there exists a point B, different from C and not an opposite of C. By

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[*] Cf. Axiom IV of Veblen's System, loc. cit., p. 345.

[†] Cf. Axiom III of Veblen's System, loc. cit., p. 344.

Theorem 1, there exists D such that CBD. By Axiom IV, CBD implies C_1DB and C_2DB. By Theorem 5 and Axiom IV, C_1DB implies C_2BD. By Axiom III, C_2BD implies DBC_2. But by Theorem 4, C_2DB implies not DBC_2. Hence the supposition that C has two opposites leads to a contradiction.

Theorem 8. If ABC, then $B \neq A$.

Proof. Suppose $B = A$. Then AAC. By Axiom III, AAC implies CAA, which is contrary to Theorem 3.

Theorem 9. If ABC, then not CAB.

Proof. By Axiom III, ABC implies CBA. By Theorem 4, CBA implies not BAC and hence, by Axiom III, not CAB.

Theorem 10. Between any two distinct, non-opposite points, there is always a third point.

Proof. Let A and B be any two points such that $A \neq B \neq A'$. By Axiom I and Theorem 7, A has one and only one opposite. If B were the opposite of A', then, by Theorems 2 and 7, B would be the same point as A, contrary to hypothesis. Hence, by Theorem 1, there exists C such that $A'BC$. By Axiom IV and Theorem 2, $A'BC$ implies ACB.

Theorem 11. If ABC, then not $AB'C$.

Proof. By Axioms III, IV, III, IV and III, $AB'C$ implies $AC'B$. By Axiom V, $AC'B$ and ABC imply $AC'C$. By Axiom III, $AC'C$ implies $CC'A$, which is contrary to the Definition of C'.

Theorem 12. If ABC and ACD, then BCD.

Proof. By Axiom III, ABC implies CBA. By Axioms III, IV and III, ACD implies CAD'. By Axiom V, CBA and CAD' imply CBD'. By Axioms III, IV and III, CBD' implies BCD.

Theorem 13. There exist infinitely many points.

Proof. By Axiom II, there exist two points A and B such that B is distinct from A and is not an opposite of A. By Theorem 1, there exists C_1 such that ABC_1. By Theorem 3, C_1 is different from A, A', B, and B'. There exists, by Theorems 1 and 3, a point C_2, different from A, A', C_1 and C_1', such that AC_1C_2. As $C_2 \neq A'$, then, by Theorems 1 and 3, there exists C_3, different from A, A', C_2 and C_2' such that AC_2C_3. By Axiom V, AC_1C_2 and AC_2C_3 imply AC_1C_3. Continue this process and obtain an infinite sequence C_1, C_2, $C_3 \cdots$. It is necessary to show that these C's are all distinct and that no two are opposites. Let us first show that AC_iC_{n+1} holds for $i = 1, 2, 3 \cdots n$. Assume it holds true for $n = k - 1$. By Axiom V, AC_iC_k and AC_kC_{k+1} imply AC_iC_{k+1}. Hence, if it holds true for $n = k - 1$, it holds true for $n = k$. But it does hold for $n = 1$. Hence our induction is complete. But, by Theorem 3, AC_iC_{n+1} implies that no C is the same as any preceding C or its opposite.

THEOREM 14. The line AB is the same as the line BA, the line $A'B$, the line $A'B'$, the line $B'A$ and the line $B'A'$.

Proof. Let X be any point of AB, to show it is also a point of the line $A'B'$. As X is on AB, $X = A$, $X = A'$, $X = B$, $X = B'$, XAB, AXB, ABX or $AX'B$. By Theorems 2 and 7, if $X = B$, then $X = (B')'$ and if $X = A$, then $X = (A')'$. By Axioms III, IV, III and IV, XAB implies $A'B'X$. By Axioms IV, III, IV, III, IV and III, AXB implies $A'X'B'$. By Axioms IV, III, IV and III, ABX implies $XA'B'$. By Axioms IV and III, Theorem 2, Axioms IV, III, IV and III, $AX'B$ implies $A'XB'$. Thus in each case X is on $A'B'$.

Similarly any point of $A'B'$ is a point of the line AB. Hence the line AB is the same as the line $A'B'$.

In like manner the line AB is the same as the line BA, the line $A'B$, the line $B'A$, and the line $B'A'$.

THEOREM 15. If a point C is on the line AB, its opposite C' is also on AB.

Proof. Suppose C is on AB. Then $C = A$, $C = B$, $C = A'$, $C = B'$, CAB, ACB, ABC or $AC'B$. By Theorem 7, if $C = A$, $C' = A'$ and if $C = B$, $C' = B'$. By Theorems 2 and 7, if $C = A'$, then $C' = A$ and if $C = B'$, then $C' = B$. By Axioms IV and III, CAB implies ABC'. By Axioms III and IV, ABC implies $C'AB$. By Theorem 2, ACB implies $A[C']'B$. Thus in each case C' is on AB.

THEOREM 16. If a point $C[A \neq C \neq A']$ is on the line AB, then B is on the line AC.

Proof. By Definition 2, C on AB implies $C = A$, $C = B$, $C = A'$, $C = B'$, CAB, ACB, ABC or $AC'B$. By hypothesis $A \neq C \neq A'$. By Theorems 2 and 7, $C = B'$ implies $B = C'$. From CAB it follows, by Axiom III, that BAC. By Axioms III, IV, III, IV and III, $AC'B$ implies $AB'C$. In each case B is on the line AC.

4. Consequences of Axioms I–V, VII.

THEOREM 17. If ABC and BCD, then D is on the line AB.

Proof. Suppose D is not on AB. Then, by Axiom VII and Theorem 10, there exists H such that CCH and DHA. But CCH is contrary to Theorem 8. Hence D is on AB.

THEOREM 18. If C and $D[C \neq D \neq C']$ are points of AB, then A is a point of CD*.

By Definition 2, C on AB implies $C = A(1)$, $C = B(2)$, $C = A'(3)$, $C = B'(4)$, $CAB(5)$, $ACB(6)$, $ABC(7)$ or $AC'B(8)$ while D on AB implies $D = A(\alpha)$, $D = B(\beta)$, $D = A'(\gamma)$, $D = B'(\delta)$, $DAB(\epsilon)$, $ADB(\zeta)$, $ABD(\eta)$, or $AD'B(\theta)$. Of the 64 conceivable cases $8[1\alpha, 1\gamma, 2\beta, 2\delta, 3\alpha, 3\gamma, 4\beta, 4\delta]$ are impossible.

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* Cf. Axiom VI of Veblen's System, loc. cit., p. 345.

We shall consider a few of the remaining 56 cases.

Case 5ϵ. CAB and DAB. By Axioms IV and III, DAB implies ABD'. By Theorem 17, CAB and ABD' imply that D' is on the line CA. By Theorem 15, as D' is on the line CA, its opposite D is on CA. By Theorem 16 as $D[C \neq D \neq C']$ is on the line CA, then A is on the line CD.

Case 6ζ. ACB and ADB. By Axioms III, IV and III, ACB implies CAB'. By Axioms III, IV, III, IV and III, ADB implies $AB'D'$. By Theorem 17, CAB' and $AB'D'$ imply that D' is on the line CA. By Theorem 15, as D' is on the line CA, its opposite D is on CA. By Theorem 16, as $D[C \neq D \neq C']$ is on the line CA, then A is on the line CD.

CASE 7ζ. ABC and ADB. By Axiom V, ADB and ABC imply that ADC. By Axiom III, ADC implies CDA. Hence, by Definition 3, A is on CD.

By similar arguments it can be shown that in all the other cases A is on CD.

THEOREM 19. If C and $D[C \neq D \neq C']$ are points of AB, then the line CD is the same as the line AB.

The proof of Theorem 19 is the same as the proof of Theorem 6 of Veblen's system.

THEOREM 20. Two distinct lines can have at most a point and its opposite in common.

5. Consequences of Axioms I–VII.

THEOREM 21. If KE be any line, there is a point not on KE.

Proof. Suppose KE contained all points. Then, by Axiom VI, it would contain three distinct points A, B and C such that A is not an opposite of B and such that C is not on the line AB. But, by Theorem 19, the line AB is the same as the line KE. But C is not on AB. Hence C is not on KE.

THEOREM 22. If ABC and BCD and $D \neq A'$, then either ABD or $AB'D$.

Proof. By Theorem 21, there exists a point P such that P is not on the line BC. By Definition 2, it follows that P is not the opposite of B. Hence, by Theorem 1, there exists O such that BPO. From BPO it follows, by Theorem 3, that $B \neq O \neq B'$. Hence were O on the line BC, by Theorem 19, the line BO would be the same as the line BC and the point P would be on BC, contrary to assumption. Hence, by Axiom VII and Theorem 10, ABC and BPO imply the existence of E such that APE

and OEC. By Axiom VII and Theorem 10, BCD and CEO imply the existence of G such that GED and BGO.

If $G = P$ then APE and PED. From APE it follows, by Theorem 3, that $P \neq E \neq P'$. From CEO it follows, by Theorem 8 and Definition 1, that $C \neq E \neq C'$ and hence were E on BC, the line CE would be the same as the line BC and O would lie on BC, which is impossible. Hence the lines PE and BC are distinct and have in common the points A and D. From ABC and BCD it follows, by Theorem 4, that $D \neq A$. By hypothesis $D \neq A'$. But, by Theorem 20, the distinct lines PE and BC can have at most a point and its opposite in common. Hence the supposition that $G = P$ leads to a contradiction. Hence $G \neq P$.

Suppose $G = P'$. Then BPO and $BP'O$. But this is contrary to Theorem 11. Hence $P \neq G \neq P'$.

Consider the triangle AED. By Theorem 10 and Axiom VII, APE and GED imply the existence of H such that GPH and AHD. By Theorem 19, the line GP is the same as the line BO. As the line AD and the line BO have the point B in common, the lines GP and AD have in common only B and B'. Then $H = B$ or $H = B'$. Therefore ABC and BCD and $D \neq A'$ imply either ABD or $AB'D$.

THEOREM 23. If ABC and ABD and $D \neq C$, then either BCD or BDC.

Proof. By Axioms III, IV and III, ABD implies BAD'. By Axiom III, ABC implies CBA. If D' were the opposite of C, then, by Theorems 2 and 7, D would be C, contrary to hypothesis. By Theorem 22, CBA and BAD' imply that either CBD' or $CB'D'$. By Axioms III, IV and III, CBD' implies BCD. By Axioms III, IV, III and IV, $CB'D'$ implies BDC.

THEOREM 24. If ABC and ABD and $D \neq C$, then either ACD or ADC.

Proof. By Axioms III and IV, ABD implies $D'AB$. If C were the opposite of D', then, by Theorems 2 and 7, C would be D, which is contrary to hypothesis. By Theorem 22, $D'AB$ and ABC imply either $D'AC$ or $D'A'C$. By Axioms IV and III, $D'AC$ implies ACD. By Axioms IV, III and IV, $D'A'C$ implies ADC.

THEOREM 25. If ABD and ACD and $B \neq C$, then either ABC or ACB.

Proof. By Axioms III, IV and III, ABD implies BAD'. By Axioms III, IV, III, IV and III, ACD implies $AD'C'$. If C' were the opposite of B, then, by Theorems 2 and 7, C would coincide with B, contrary to hypothesis. By Theorem 22, BAD' and $AD'C'$ imply that either BAC' or $BA'C'$. By Axioms III, IV and III, BAC' implies ABC. By Axioms III, IV, III and IV, $BA'C'$ implies ACB.

Theorem 26. If ABC is a triangle and E and F are points such that ABE and AFC, then there exists G such that FGE and BGC.

Proof. By Axioms III and IV, ABE implies $E'AB$ and hence, by Axiom III, BAE'. By Theorem 10 and Axiom VII, BAE' and AFC imply the existence of G such that BGC and $E'FG$. By Axioms IV and III, $E'FG$ implies FGE.

Theorem 27. If ABC is a triangle and E and F are points such that AFB and AEC, then there exists D such that BCD and FED.

Proof. By Axioms III and IV, AFB implies $B'AF$. By Axioms VII and Theorem 10, $B'AF$, AEC and C not on AB (and hence not on $B'A$, by Theorem 14) imply the existence of D such that FED and $B'DC$. By Axiom IV, $B'DC$ implies BCD.

Theorem 28. A line meeting two sides of a triangle cannot meet the third side.

Proof. Let ABC be a triangle and suppose the line l meets the side AB in F and the side AC in E. Then AFB and AEC. By Theorem 27, there exists D such that BCD and FED. From FED it follows, by Theorem 8 and Definition 1, that $F \mp E \mp F'$ and hence, by Theorem 19, that the line l is the same as the line FE. By Theorems 15 and 20, AB and BC have only B and B' in common. Hence, were F on BC, F would be either B or B'.

If $F = B$, then ABB which is contrary to Theorem 3, while if $F = B'$ then $AB'B$, contrary to Theorem 3. Hence F is not on BC.

As the distinct lines FE and BC have the point D in common, they can have only D and D' in common, by Theorems 15 and 20. Hence if l, which is the same as the line FE, meets the side BC, it must meet it in either D or D'. Hence BDC or $BD'C$.

But from BCD it follows, by Axiom III and Theorem 9, that BDC is false and, by Theorem 11 and Axioms III, IV, III, IV and III, that $BD'C$ is false.

Theorem 29. If O is a point in the interior of the triangle ABC, then there exists Q such that AOQ and BQC.

Proof. Case I. Suppose O is between M and N where M is a point of the side AB and N is a point of the side BC. Then AMB, MON and BNC. From BNC it follows, by Theorem 8 and Definition 1, that $B \mp N \mp B'$ and hence, by Theorem 19, that the line BN is the same as the line BC. By Theorems 15 and 20, the lines AB and BN have in common only B and B'. Hence were M on BN, M would be B or B' which, by Theorem 3, is contrary to AMB. By Axiom VII, AMB, MON and M not on BN imply the existence of Q such that AOQ and BQN. By Axiom V, BQN and BNC imply BQC.

Case II. Suppose O is between M of the side AB and N of the side AC. Then AMB, MON and ANC. By Theorem 3, it follows from AMB that $M \neq B \neq M'$ and hence, by Theorem 19, that the line BM is the same as the line AB. Therefore, by Theorem 15 and 20, the lines BM and AC can have in common only A and A'. Hence were N on BM, N would be A or A', which, by Theorem 8 and Definition 1, is contrary to ANC. In like manner it follows that N is not on BC. Consider first the triangle BMN. By Axiom VII and Theorem 10, AMB and MON imply the existence of E such that BEN and AOE. Consider next the triangle BNC. By Axiom VII and Theorem 10, ANC and BEN imply the existence of Q such that AEQ and BQC. By Axiom V, AOE and AEQ imply AOQ.

Theorem 30. A point and its opposite do not both belong to the interior of the same triangle.

Proof. Suppose P and P' are interior points of the triangle ABC. Then, by Theorem 29, there exist points E and F, such that APE, BEC, $AP'F$ and BFC. By Axioms III, IV, III, IV and III, $AP'F$ implies $AF'P$. By Theorem 20, AP and BC have in common only E and E'. Hence either F is E or F is E'.

Case I. Suppose $F = E$. Then APE and $AP'E$, which is contrary to Theorem 11.

Case II. Suppose $F = E'$. Then BEC and $BE'C$, which is contrary to Theorem 11.

Theorem 31. If ABC is a triangle and X and F are such that AXF and BFC, then X is within the triangle ABC.

Proof. By Theorem 10, there exists E such that CEF. By Axiom III, BFC implies CFB. By Axiom V and Theorem 12, CEF and CFB imply CEB and EFB. By Theorem 3, CFB implies that $F \neq B \neq F'$ and hence, by Theorem 19, that the line BF is the same as the line BC. As A is not on BC, by hypothesis, it cannot be on BF. Consider the triangle AFB. By Axiom VII, EFB and AXF imply the existence of K such that KXE and AKB. By Definition 4, AKB, KXE and BEC imply that X is within the triangle ABC.

Theorem 32. If X is within the triangle ABC, then X' is within the triangle $A'B'C'$.

Proof. As X is within the triangle ABC, it follows, by Theorem 29, that there exists P such that AXP and BPC. By Axioms IV, III, IV; III, IV and III, AXP and BPC imply $A'X'P'$ and $B'P'C'$ respectively. Hence, by Theorem 31, X' is within the triangle $A'B'C'$.

Theorem 33. If O is a point collinear with 2 points M and N of a triangle ABC, the line joining O to any point $P[O \neq P \neq O']$ of a side of the triangle, meets the perimeter in a point $Q[Q \neq P]$.

THEOREM 34. If A, B and C are any three non-collinear points of the plane DEF, then the plane DEF is the same as the plane ABC.

Theorems 33 and 34 are the same as Veblen's Theorems 15 and 16. His proofs, with slight modifications for opposites, hold good here.

THEOREM 35. If two distinct non-opposite points lie in a plane, the line joining them lies in that plane.

Proof. Suppose A and B are distinct non-opposite points lying in the plane α. By Definition 5 and Theorem 19, there exists in the plane α a point C such that A, B and C are non-collinear. But, by Theorem 34, the plane ABC is the same as the plane α. The line AB is in the plane ABC. Hence AB is in α.

THEOREM 36. Every two lines in a plane intersect in two points, which are each other's opposites.

Proof. Let AB and CD be two lines in the same plane α and let us suppose they have no common point.

If B were D', then B would be on CD, contrary to the assumption that AB and CD have no common point. As $B \neq D'$ it follows, by Theorem 1, that there exists E such that DBE. From DBE it follows, by Theorem 3, that $D \neq E \neq D'$ and hence, if E were on CD, by Theorem 19, the lines DE and CD would be identical. Then B would lie on both AB and CD, contrary to our assumption concerning AB and CD. In like manner E is not on AB.

By Theorem 34, C, D and E, three non-collinear points of the plane α, determine the plane. By Theorem 33, the line AB must meet the perimeter of the triangle CED in a point $Q[Q \neq B]$.

As E is not on AB, Q cannot be E. By hypothesis, AB and CD have no common point. Hence Q cannot be C, D or a point of the side CD.

By Theorems 15 and 20, AB and ED have in common only B and B'. Hence were Q a point of the side ED, Q would be B'. But it follows from DBE, by Theorem 11 and Axiom III, that $EB'D$ is false. Hence, by Definitions 2 and 3, Q is not a point of the side ED.

Therefore Q must be a point of the side EC. By Theorem 27, EBD and EQC imply the existence of H such that BQH and DCH. By Axiom III, DCH implies HCD. By Theorem 8 and Definition 2, BQH implies $B \neq Q \neq B'$ and hence, by Theorem 19, that the line AB is the same as the line BQ. Therefore the lines AB and CD have H in common. Therefore, by Theorem 15, they have H' in common.

6. Consequences of Axioms I–IX.

THEOREM 37. If KEF be any plane, there is a point not on KEF.

Proof. Suppose the plane KEF contained all points. Then, by Axiom

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* Loc. cit., page 359.

VIII, the plane would contain three non-collinear points A, B and C, and a point D such that D is not on the plane ABC. But, by Theorem 34, the plane KEF is the same as the plane ABC. But D, which is not on ABC, is not on KEF.

Definition 8. A point X is *interior to the tetrahedron* $ABCD$ if it lies between two points F and G of different faces.

Theorem 38. If a vertex A of a tetrahedron $ABCD$ be joined to an interior point O of the tetrahedron, the line AO meets the opposite face BCD in a point P such that AOP.

The proof of Theorem 38 is exactly the same as the proof of Theorem 19 of Veblen's system.*

Theorem 39. A point and its opposite do not lie within the same tetrahedron.

Proof. Suppose the point O and its opposite O' lie within the same tetrahedron $ABCD$. By Theorem 38, there exist F and G in the face BCD such that AOF and $AO'G$. By Axioms III, IV, III, IV and III, $AO'G$ implies $AG'O$. Hence F and G are both points of the line AO. The point F must then be either G or G'. For were $F \neq G \neq F'$, the line FG would be the same as the line AO, by Theorem 19. But F and G lie in the plane BCD. Hence, by Theorem 35, every point of FG lies in the plane BCD and thus A would lie in BCD, contrary to hypothesis.

Case I. Suppose $F = G'$. Then G and G' are both within the triangle BCD, contrary to Theorem 30.

Case II. Suppose $F = G$. Then AOG and $AO'G$, contrary to Theorem 11.

Theorem 40. If any interior point O of a tetrahedron $ABCD$ be joined to another point P of the surface or interior of the tetrahedron, the line OP meets the surface in two points Q and R such that QOR.

Theorem 41. If O is a point in the order MNO with two points M and N of different faces of the tetrahedron $ABCD$, the line joining O to an interior point P of a face of the tetrahedron, meets the surface in Q such that $Q \neq P \neq Q'$.

Theorem 42. If two points lie in a space $ABCD$, so does every point of the line joining them.

Theorem 43. Any four non-coplanar points of a space, determine *that* space.

Theorem 44. There is but one space.

Theorem 45. If two planes have a point O in common, they have in common a second point $P[P \neq O \neq P']$.

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* Loc. cit., p. 361.

Theorems 40–45 are the same as Veblen's Theorems 20–25. His proofs with slight modifications hold good here.*

Theorem 46. Every two planes in space intersect in a straight line.

Proof. Let α and β be any two planes and let us suppose they have no common point.

Take A any point of α and B any point of β. The point B must be different from A', for otherwise B would lie on α, contrary to hypothesis. By Theorem 1, there exists C such that ABC. By Theorem 3, it follows from ABC that $A \mp C \mp A'$ and that $B \mp C \mp B'$. As the point B is not on the plane α, it follows, by Theorem 35, that C cannot be on α. In like manner C is not on β.

Take D any point in the plane β such that $D \mp B \mp D'$. The point D is not on the line CB. For were D on CB, by Theorem 16, C would be a point of BD. As B and $D(B \mp D \mp B')$ are on β every point of the line BD is on β. Hence C would lie on β, which is impossible. The points C, B, and D are non-collinear and determine a plane γ.

As the points B and C are in the plane γ, by Theorem 34, A is a point of γ. Hence the planes α and γ have one common point. Therefore, by Theorem 45, they must have a second common point E such that $A \mp E \mp A'$.

By Theorem 36, the lines EA and DB, both lying in the plane γ, must intersect in some point R. As E and A both lie in the plane α, by Theorem 33, the point R must lie in α. In like manner the point R must lie in β. As the planes α and β have one common point, by Theorem 45, they must have a second point S in common such that $S \mp R \mp S'$. By Theorem 35, the line SR lies in both α and β.

7. The Introduction of Congruence.

Definition. The pairs of opposite points and the planes of space constitute a *polar system* if they are in such a reciprocal one-to-one correspondence that to the $\left\{\begin{matrix}\text{pair of opposite points}\\ \text{plane}\end{matrix}\right\}$ incident with three $\left\{\begin{matrix}\text{non-collinear planes}\\ \text{non-collinear pairs of opposite points}\end{matrix}\right\}$ corresponds the $\left\{\begin{matrix}\text{plane}\\ \text{pair of opposite points}\end{matrix}\right\}$ incident with the corresponding $\left\{\begin{matrix}\text{pairs of opposite points}\\ \text{planes}\end{matrix}\right\}$. A pair of opposite points are the *poles* of the corresponding plane, while a plane is the *polar* of its corresponding pair of opposite points. A polar system is *elliptic* if no $\left\{\begin{matrix}\text{pair of opposite points}\\ \text{plane}\end{matrix}\right\}$ is on its corresponding $\left\{\begin{matrix}\text{plane}\\ \text{pair of opposite points}\end{matrix}\right\}$.

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* Loc. cit., pp. 361–363.

That our hypotheses are sufficient for the proof of the existence of elliptic polar systems can easily be verified.* Pick out a definite elliptic polar system Σ.

The segment AB is said to be congruent to the segment CD, if AB can be carried into CD by a projective transformation leaving Σ invariant.

Definition. If A and B are two distinct, non-opposite points the half line AB is made up of the set of all points $[X]$ such that $X = B$, AXB or ABX.

Definition. If A, B, and C are non-collinear, the boundary of the angle BAC consists of A and the half lines AB and AC.

Two angles are said to be *congruent* if they are such that the sides of one can be transformed into the sides of the other by a projective transformation leaving Σ invariant.

The usual properties of congruent segments and congruent angles can easily be deduced.†

It is now clear that the points of our space can be brought into one-to-one correspondence with the points of the hypersphere $x^2 + y^2 + z^2 + w^2 = r^2$ in such a manner that the three points A, B and C are in the order ABC, if, and only if, the corresponding points A', B' and C' on the hypersphere are such that B' is on the shorter arc of the geodesic on the hypersphere from A' to C'.

It follows that our set of axioms is categorical.

8. Questions of Independence.

Let G denote the set of Axioms I–X.

In the following E_i denotes an example of a system in which Axiom i is false but all the other Axioms of the set G are true. In every E_i use is made of a well-defined space S_i.

E_1. S_1 is a Euclidean space of three dimensions. Order has its usual significance.

E_2. S_2 consists of two points A and A'. No order exists.

E_3. S_3 consists of three points A, B and C in the orders AAC, ACC, BCA, BCC.

E_4. S_4 consists of three points A, B and C in the orders ABC and CBA.

E_5. S_5 consists of four points A, A', B and B' in the orders AAB, $A'BA$, ABA', $A'A'B$, $BA'A'$, $B'A'A'$, BAA, $B'AA$, AAB', $A'B'A$, $AB'A'$, $A'A'B'$.

E_6. S_6 consists of the points of the circumference of a circle. Three distinct points A, B and C are said to be in the order ABC, if, and only if, B is on the shorter arc from A to C.

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* See J. Steiner, Synthetische Geometrie, pp. 411–414 and 422–424.

† For a set of congruence axioms for Double Elliptic Geometry of Two Dimensions see G. B. Halsted, loc. cit., pp. 217–19.

E_7. The points of S_7 are the points on a hypersphere in space of four dimensions. Pick out a definite line l_1 and one pair of opposite points O and O' on l_1.

If the points A_1, A_2 and A_3 are not all on the line l_1, then they are said to be in the *order* $A_1A_2A_3$, if, and only if, A_2 is on the shorter arc of the geodesic from A_1 to A_3. If A_1, A_2 and A_3 are all on the line l, then they are said to be in the order $A_1A_2A_3$, if, and only if, $\bar{A}_2$ is on the shorter arc of the geodesic from $\bar{A}_1$ to $\bar{A}_3$, where (1) $\bar{A}_i$ is A_i, if A_i is neither O nor O', (2) $\bar{A}_i$ is O', if A_i is O, (3) $\bar{A}_i$ is O, if A_i is O'.

E_8. S_8 consists of the points of the surface of a sphere. Three distinct points A, B and C are said to be in the order ABC, if, and only if, B is on the shorter arc of the great circle from A to C.

E_9. S_9 is a four-dimensional double elliptic space.

E_{10}. S_{10} is Rational Three-Dimensional Double Elliptic Geometry.

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